

Entropy Product Formula for Gravitational Instanton

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Abstract

We investigate the entropy product formula for various gravitational instantons. We speculate that due to the mass-independent characteristics of the said instantons they are *universal* as well as they are *quantized*. For isolated Euclidean Schwarzschild BH, these properties simply *fail*.

1 Introduction

There has been a strong interest in microscopic interpretation of black hole (BH) entropy [1, 2, 3, 4, 5, 6, 7, 8] in terms of D -branes come due to the work by Strominger and Vafa [9]. In d -dimension Euclidean quantum gravity, this entropy is due to the $(d - 2)$ -dimensional fixed point sets of the imaginary time translation Killing vector. There are many fixed point sets which could also give rise to BH entropy.

Previously, Area (or Entropy) product formula evaluated for different class of BHs [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. In some cases, the product formula is not mass-independent (universal) and in some cases the product formula is indeed mass-independent i.e. universal. Upto the author's knowledge, there has been no attempt to compute the entropy product formula for *gravitational instanton*.

Thus in the present work, we wish to investigate the entropy product formula for various gravitational instantons. Instantons are non-singular and having imaginary time. They arises in quantum field thory in evaluating the functional integral, in which the functional integral is Wick rotated and expressed as an integral over Euclidean field configurations. They are the solutions of the Euclidean Einstein equations. They have the signature of the form $(+ + + +)$. There are two types of instantons discovered so far. One is an asymptotically locally flat(ALF), which was first discovered by Hawking in 1977 [29] and the other is an asymptotically locally Euclidean(ALE), discovered by Gibbons and

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Hawking in 1978 [30]. The examples of ALE classes are Flat space, Eguchi-Hanson [31] and multi-instanton [30].

The ALF class of solutions are asymptotically flat in the 3D sense, the fourth, imaginary -time, direction being periodic. The surfaces of large radii could be think of as an \mathcal{S}^1 bundle over \mathcal{S}^2 . The product bundle corresponds to the asymptotically flat(AF) solutions which include the Euclidean Schwarzschild and Euclidean Kerr solutions [32, 33]. The twisted bundles correspond to the multi-Taub-NUT solution [29], and the Taub-Bolt solution first discovered by Page [34].

In our previous work [27, 28, 18], we investigated the properties of inner and outer horizon thermodynamics of Taub-NUT (Newman-Unti-Tamburino) BH, Kerr-Taub-NUT BH and Kerr-Newman-Taub-NUT BH in four dimensional Lorentzian geometry. The failure of *First law* of BH thermodynamics and *Smarr-Gibbs-Duhem* relation for Taub-NUT and Kerr-Taub-NUT BH in the Lorentzian regime gives the motivation behind this work. What happens one can go from Lorentzian geometry to Riemannian geometry? This is the prime aim in this work. By studying the properties of these instantons what would be the effects on the BH entropy product formula due to the non-trivial NUT parameter?

In general relativity, the non-trivial value for the BH entropy is due to the presence of the fixed point set of the periodic imaginary time Killing vector. The fixed point set here we considered actually is the BH horizons (\mathcal{H}^\pm). Here \mathcal{H}^+ is called event horizon and \mathcal{H}^- is called the Cauchy horizon. In four dimension, such fixed point sets are of two types, isolated points or zero dimensional which we call NUTs and two surfaces or two dimensional which we call Bolts. Thus one can thought Bolts as being the analogue of electric type mass- monopoles and the NUTs as being gravitational dyons endowed with a real electric type mass-monopole and an imaginary magnetic type mass-monopole. The presence of magnetic type mass introduces a Dirac string like singularity in the spacetime which is so called Misner string was first pointed out by Misner in his paper on the Lorentzian Taub-NUT spacetime [35]. A Misner string is a coordinate singularity which can be considered as a manifestation of a “non-trivial topological twisting” [36] of the manifold (M, g_{ab}) . This twist is parametrized by a topological term, the NUT charge.

In our previous investigation [18], we have taken the metric in a Lorentzian spacetime in $3 + 1$ split form as

$$ds^2 = -\mathcal{F} (dt + w_i dx^i)^2 + \frac{\gamma_{ij}}{\mathcal{F}} dx^i dx^j . \quad (1)$$

In this work, we are interested to study the metric in a *Riemannian* spacetime which could be written in $3 + 1$ split form:

$$ds^2 = \mathcal{F} (d\tau + w_i dx^i)^2 + \frac{\gamma_{ij}}{\mathcal{F}} dx^i dx^j . \quad (2)$$

Here all quantities are independent of t or τ . The Wick rotation that transforms from one case to other is by the transformation $t \mapsto i\tau$ and $w_i \mapsto iw_i$. \mathcal{F} can be thought of as an

electric type potential, and ω_i or w_i as a magnetic-type vector potential. The associated magnetic field is $H_{ij} = \partial_i \omega_j - \partial_j \omega_i$ and it is gauge invariant. This could be used to define a magnetic monopole moment called the NUT charge n [30]. If $n \neq 0$, the fibration could not be trivial. In the Lorentzian geometry, these fixed point sets are the two-dimensional Boyer bifurcation sets of event horizon [5, 37]. In Riemannian geometry, these fixed point sets are of two types: zero dimensional point or NUTs, and two dimensional surfaces or Bolts [30]. A NUT possesses a pair of surface gravities κ_1 and κ_2 . p and q are a pair of coprime integers such that $\frac{\kappa_1}{\kappa_2} = \frac{p}{q}$. If $\frac{\kappa_1}{\kappa_2}$ is irrational, $p = q = 1$. A NUT of type (p, q) has a NUT charge of $n = \frac{\beta}{8\pi pq}$. β is the period of the imaginary time coordinate.

The structure of the paper is as follows. In Sec. 2, we have considered the Euclidean Schwarzschild BH. In Sec. 3, we have investigated the properties of Self dual Taub-NUT instantons. In Sec. 4, we have studied the mass-independent properties of Taub-Bolt instantons. In Sec. 5, we have described the properties of Eguchi-Hanson instantons. In Sec. 6, we have examined the properties of Taub-NUT-AdS Spacetime. In Sec. 7, we have studied the entropy product formula for Taub-Bolt-AdS spacetime and finally in Sec. 8, we have examined the product rules for Dyonic Taub-NUT-AdS and Taub-Bolt-AdS spacetime.

2 Euclidean Schwarzschild metric:

To give a warm up, let us consider the Schwarzschild BH (where $G = c = 1$) in Euclidean form as

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) . \quad (3)$$

The apparent singularity at the event horizon $r_+ = 2M$ can be removed by identifying τ with a period $\beta = 8\pi M$ [33, 29]. The radial coordinate has the range $2M \leq r \leq \infty$. Then the topology of the manifold is $R^2 \times \mathcal{S}^2$. The isometry group is $O(2) \otimes O(3)$, where the $O(2)$ corresponds to translations in the periodically identified imaginary time τ and the $O(3)$ corresponds to rotations of the θ and ϕ coordinates.

The Killing vector ∂_τ has unit magnitude at large radius and has a Bolt on the horizon $r_+ = 2M$ which is a 2-sphere \mathcal{S}^2 of area

$$A_+ = 16\pi M^2 . \quad (4)$$

The surface gravity is given by

$$\kappa_+ = \frac{2\pi}{\beta} = \frac{1}{4M} . \quad (5)$$

Thus the BH temperature is

$$T_+ = \frac{\kappa_+}{2\pi} = \frac{1}{8\pi M} . \quad (6)$$

Thus for an isolated Euclidean Schwarzschild BH the area product would be

$$A_+ = 16\pi M^2 . \quad (7)$$

which tells us that the product is dependent on mass and thus it is not universal. Also it is not quantized. The Euclidean action derived in [29]

$$I = -\ln Z = 4\pi M^2 . \quad (8)$$

$$(9)$$

From that one could derived the entropy as in [29]

$$S_+ = -\left(\beta \frac{\partial}{\partial \beta} - 1\right) \ln Z = 4\pi M^2 . \quad (10)$$

Thus the entropy product for isolated Euclidean Schwarzschild BH should be

$$S_+ = 4\pi M^2 . \quad (11)$$

Indeed, it is not universal nor it is quantized.

3 Self dual Taub-NUT Instantons:

In this section we shall calculate the entropy product and area product of four dimensional Taub-NUT spacetime. It is of ALF. ALF metrics have a NUT charge, or magnetic type mass, n , as well as the ordinary electric type mass, M . The NUT charge is $\frac{\beta c_1}{8\pi}$, where c_1 is the first Chern number of the $U(1)$ bundle over the sphere at infinity, in the orbit space Ξ . If $c_1 = 0$, then the boundary at infinity is $\mathcal{S}^1 \times \mathcal{S}^2$ and the spacetime is AF. The BH metrics are saddle points in the path integral for the partition function. Thus, if $c_1 \neq 0$, the boundary at infinity is a squashed \mathcal{S}^3 , and the metric could not be analytically continued to a Lorentzian metric. The squashed \mathcal{S}^3 is the three-dimensional space on which the boundary conformal field theory will be compactified, with β identified with the inverse temperature i.e. $T = 1/\beta$.

Hawking [29] first given an examples of gravitational instanton was the self-dual Taub-NUT metric described by

$$ds^2 = \mathcal{F}(r) (d\tau + 2n \cos \theta d\phi)^2 + \frac{dr^2}{\mathcal{F}(r)} + (r^2 - n^2) (d\theta^2 + \sin^2 \theta d\phi^2) . \quad (12)$$

$$\mathcal{F}(r) = \frac{r - n}{r + n} \quad (13)$$

It is ALF with a central NUT. The self-dual Taub-NUT instanton has $M = n$ and the anti-self-dual instanton has $M = -n$. The value $r = n$ is now a zero in $\mathcal{F}(r)$. The (θ, ϕ)

two-sphere has a zero area at $r = n$, so the zero in $\mathcal{F}(r)$ is a zero-dimensional fixed point of ∂_ϕ , a NUT.

In order to make the solution regular, we take the region $r \geq n$ and let the period of τ be $8\pi n$. The metric has a NUT at $r = n$, with a Misner string running along the z -axis from the NUT out to infinity i.e. $n \leq r \leq \infty$.

We know from the idea of path-integral formulation of quantum gravity which tells us that the Euclidean action derived in [29]

$$I = -\ln Z = 4\pi n^2 . \quad (14)$$

where Z is the partition function of an ensemble

$$Z = \int [Dg][D\phi] e^{-I(g,\phi)} . \quad (15)$$

with the path integral taken over all metrics g and matter field ϕ that are appropriately identified with the period β of τ . Therefore the entropy could be derived as

$$S = -\left(\beta \frac{\partial}{\partial \beta} - 1\right) \ln Z = 4\pi n^2 . \quad (16)$$

It is indeed mass independent and thus it is universal.

The surface gravity is calculated to be

$$\kappa = \frac{2\pi}{\beta} = \frac{1}{4n} . \quad (17)$$

Thus the BH temperature should read off

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi n} . \quad (18)$$

Now we see what happens the above results for another instantons that is Taub-Bolt.

4 Taub-Bolt Instantons:

The Taub-Bolt instanton is described by the metric [34]

$$ds^2 = \mathcal{G}(r) (d\tau + 2n \cos \theta d\phi)^2 + \frac{dr^2}{\mathcal{G}(r)} + (r^2 - n^2) (d\theta^2 + \sin^2 \theta d\phi^2) . \quad (19)$$

$$\mathcal{G}(r) = \frac{(r - 2n)(r - \frac{n}{2})}{(r^2 - n^2)} = \frac{(r - r_+)(r - r_-)}{(r + n)(r - n)} \quad (20)$$

It is a non-self-dual, non-compact solution of the vacuum Euclidean Einstein equations. In order to make the solution regular we have restricted in the region $r = r_+ \geq 2n$, and

the Euclidean time has period $\beta = 8\pi n$. Asymptotically, the Taub-Bolt instanton behaves similar manner as the Taub-NUT, so it is ALF. Since we are setting the fixed point is at $r = r_+ = 2n$, therefore the area of the \mathcal{S}^2 does not vanish there and the fixed point set is 2-dimensional, thus it is a Bolt of area

$$A_+ = 12\pi n^2 . \quad (21)$$

Thus the area product for Taub-Bolt instanton will be

$$A_+ = 12\pi n^2 . \quad (22)$$

Thus the area product does independent of mass and also quantized. Similarly, the action was calculated in [38]

$$I = -\ln Z = \pi n^2 . \quad (23)$$

Thus the entropy was derived by the universal formula

$$S_+ = -\left(\beta \frac{\partial}{\partial \beta} - 1\right) \ln Z = \pi n^2 . \quad (24)$$

5 Eguchi-Hanson Instantons:

A non-compact instanton which is a limiting case of the Taub-NUT solution is the Eguchi-Hanson metric [31],

$$\begin{aligned} ds^2 = & \left(1 - \frac{n^4}{r^4}\right) \left(\frac{r}{8n}\right)^2 (d\tau + 4n \cos \theta d\phi)^2 + \frac{dr^2}{\left(1 - \frac{n^4}{r^4}\right)} \\ & + \frac{r^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2) . \end{aligned} \quad (25)$$

The instanton is regular if we consider the region $r \geq n$, and let τ has period $8\pi n$. The metric is ALE type. There is a Bolt of area at $r = n$ is given by

$$A = \pi n^2 . \quad (26)$$

which gives rise to a Misner string along the z -axis. Thus the product is universal and could be quantized. The Euclidean action was derived in [38]

$$I = 0 . \quad (27)$$

Thus entropy corresponds to

$$S = \left(\beta \frac{\partial}{\partial \beta} - 1\right) I = 0 . \quad (28)$$

We now turn our attention for the Taub-NUT and Taub-Bolt geometries in four dimensional locally AdS spacetime. The spacetimes have a global non-trivial topology due to the fact that one of the Killing vector has a zero dimensional fixed point set called NUT or a two-dimensional fixed point set called Bolt. Moreover, these four dimensional spacetimes have Euclidean sections which can not be exactly matched to AdS spacetime at infinity.

6 Taub-NUT-AdS Spacetime:

In this section we shall consider the spacetime which are only locally asymptotically AdS and with non-trivial topology. The metric on the Euclidean section of this family of solutions could be written as [39, 40]

$$ds^2 = \mathcal{H}(r) (d\tau + 2n \cos \theta d\phi)^2 + \frac{dr^2}{\mathcal{H}(r)} + (r^2 - n^2) (d\theta^2 + \sin^2 \theta d\phi^2) . \quad (29)$$

where

$$\mathcal{H}(r) = \frac{(r^2 + n^2) - 2Mr + \ell^{-2} (r^4 - 6n^2 r^2 - 3n^4)}{r^2 - n^2} \quad (30)$$

and $\ell^2 = \frac{-3}{\Lambda}$, with $\Lambda < 0$ being the cosmological constant. Here M is a (generalized) mass parameter and r is a radial coordinate. Also, τ , the analytically continued time i.e. Euclidean time, parametrizes a circle \mathcal{S}^1 , which is fibered over the two-sphere \mathcal{S}^2 , with coordinates θ, ϕ . The non-trivial fibration is a consequence of a non-vanishing NUT parameter n .

There are some restrictions [41] for existence of a regular NUT parameter. Firstly, in order to ensure that the fixed point set is zero dimensional, it is necessary that the Killing vector ∂_τ has a fixed point which occurs precisely when the area of the two-sphere is zero size. Secondly, in order for the Dirac-Misner string [35] to be unobservable, it is necessary that the period of τ be $\beta = 8\pi n$. To avoid the conical singularity, we must check $\mathcal{H}'(r_+ = n) = \frac{1}{2n}$. Thirdly, the mass parameter M must be $M = n - \frac{4n^3}{\ell^2}$. After simplifying the metric coefficients, we obtain

$$\mathcal{H}(r) = \left(\frac{r-n}{r+n} \right) \left[1 + \frac{(r-n)(r+3n)}{\ell^2} \right] \quad (31)$$

and the range of the radial coordinate becomes $n \leq r \leq \infty$. For our requirement, the Euclidean action for this spacetime was calculated in [42, 43]

$$I = -\ln Z = 4\pi n^2 \left(1 - \frac{2n^2}{\ell^2} \right) . \quad (32)$$

and the entropy will be

$$S_+ = \left(\beta \frac{\partial}{\partial \beta} - 1 \right) I = 4\pi n^2 \left(1 - \frac{6n^2}{\ell^2} \right) . \quad (33)$$

Thus the entropy product should be

$$S_+ = 4\pi n^2 \left(1 - \frac{6n^2}{\ell^2} \right) . \quad (34)$$

It is independence of mass parameter and does depend on NUT parameter and cosmological constant. Thus the entropy product is universal for Taub-NUT-Ads spacetime. The Hawking temperature $T_+ = \frac{1}{8\pi n}$ is same as Taub-NUT BH. The first law of thermodynamics is also satisfied as $dM = T_+ dS$.

7 Taub-Bolt-AdS Spacetime:

For Taub-Bolt-AdS, the metric has the same form as in (29) but the fixed point set here is two dimensional or Bolt and with additional restrictions are the metric coefficients $\mathcal{H}(r)$ vanish at $r = r_b > n$. In order to have a regular Bolt at $r = r_b$, the following conditions must be satisfied: (i) $\mathcal{H}(r_b) = 0$, (ii) $\mathcal{H}'(r_b) = \frac{1}{2n}$ and the numerator of $\mathcal{H}(r)$ at $r = r_b$ being a single one. From the condition (i), we get the mass parameter at $r = r_b$:

$$M = M_b = \frac{r_b^2 + n^2}{2r_b} + \frac{1}{2\ell^2} \left(r_b^3 - 6n^2 r_b - 3\frac{n^4}{r_b} \right) . \quad (35)$$

Then we find[41]

$$\mathcal{H}'(r_b) = \frac{3}{\ell^2} \left(\frac{r_b^2 - n^2 + \ell^2/3}{r_b} \right) \quad (36)$$

To satisfy the condition (ii) we must have the quadratic equation for r_b :

$$6nr_b^2 - \ell^2 r_b - 6n^3 + 2n\ell^2 = 0 . \quad (37)$$

which gives the solution for r_b in two branches

$$r_{b\pm} = \frac{\ell^2}{12n} \left(1 \pm \sqrt{1 - 48\frac{n^2}{\ell^2} + 144\frac{n^4}{\ell^4}} \right) . \quad (38)$$

The discriminant of the above equation must be negative for r_b to be real and for $r_b > n$ we obtain the following inequality for n :

$$n \leq n_{max} = \sqrt{\frac{1}{6} - \frac{\sqrt{3}}{12}} \ell . \quad (39)$$

The Euclidean action was computed in [43]

$$I = \frac{4\pi n}{\ell^2} (M_b \ell^2 + 3n^2 r_b - r_b^3) . \quad (40)$$

Now it can be easily derive the entropy via the universal entropy formula:

$$S_+ = \left(\beta \frac{\partial}{\partial \beta} - 1 \right) I = 4\pi n \left(M_b - \frac{3n^2 r_b}{\ell^2} + \frac{r_b^3}{\ell^2} \right) . \quad (41)$$

8 Dyonic Taub-NUT-AdS and Taub-Bolt-AdS Spacetime:

The general form of the metric for dyonic Taub-NUT-AdS spacetime [44, 45, 46] is given by

$$ds^2 = \mathcal{N}(r) (d\tau + 2n \cos \theta d\phi)^2 + \frac{dr^2}{\mathcal{N}(r)} + (r^2 - n^2) (d\theta^2 + \sin^2 \theta d\phi^2) . \quad (42)$$

where,

$$\mathcal{N}(r) = \frac{(r^2 + n^2 + 4n^2 \nu^2 - q^2) - 2Mr + \ell^{-2} (r^4 - 6n^2 r^2 - 3n^4)}{r^2 - n^2} \quad (43)$$

The gauge field reads off

$$A \equiv A_\mu dx^\mu = \left(\frac{qr}{r^2 - n^2} + \nu \frac{r^2 + n^2}{r^2 - n^2} \right) (d\tau - 2n \cos \theta d\phi) , \quad (44)$$

The conditions of smoothness of the Euclidean section implies that the parameter q is related to the parameter ν gives a deformation from the uncharged system. When these parameters go to zero value, we obtain simply Taub-NUT-AdS spacetime.

In order to have a regular position of NUT or Bolt at $r = r_\pm$, we set $\mathcal{N}(r) = 0$ and also the gauge field A must be regular at that point. Thus we obtain the mass parameter as

$$M = \frac{r_\pm^2 + n^2 + 4n^2 \nu^2 - \nu^2}{2r_\pm} + \frac{1}{2\ell^2} \left(r_\pm^3 - 6n^2 r_\pm - 3\frac{n^4}{r_\pm} \right) . \quad (45)$$

and

$$q = -\frac{r_\pm^2 + n^2}{r_\pm} \nu . \quad (46)$$

The electric charge and potential at infinity corresponds to

$$Q = q \phi_{\pm} = -q \frac{r_{\pm}}{r_{\pm}^2 + n^2} = \nu . \quad (47)$$

Now the Euclidean action for the above spacetime was calculated in [47] (in units where $G = c = 1$)

$$I_{\pm} = -2\pi \frac{[r_{\pm}^4 - \ell^2 r_{\pm}^2 + n^2 (3n^2 - \ell^2)] r_{\pm}^2 - (r_{\pm}^4 + 4n^2 r_{\pm}^2 - n^4) \ell^2 \nu^2}{(3r_{\pm}^2 - 3n^2 + \ell^2) r_{\pm}^2 + (r_{\pm}^2 - n^2) \ell^2 \nu^2} . \quad (48)$$

The entropy was calculated as:

$$S_{\pm} = 2\pi \frac{[3r_{\pm}^4 + (\ell^2 - 12n^2) r_{\pm}^2 + n^2 (\ell^2 - 3n^2)] r_{\pm}^2 + (r_{\pm}^4 + 4n^2 r_{\pm}^2 - n^4) \ell^2 \nu^2}{(3r_{\pm}^2 - 3n^2 + \ell^2) r_{\pm}^2 + (r_{\pm}^2 - n^2) \ell^2 \nu^2} . \quad (49)$$

When we set $r_{\pm} = r_n = n$, we get a dyonic NUT spacetime. For this spacetime the above calculations reduced to

$$M = n - \frac{4n^3}{\ell^2}, \quad Q = -2n\nu, \quad \phi_{\pm} = \nu . \quad (50)$$

and

$$I_{\pm} = 4\pi n^2 \left(1 - 2\frac{n^2}{\ell^2} + 2\nu^2 \right) . \quad (51)$$

Finally, the entropy is given by

$$S_{\pm} = 4\pi n^2 \left(1 - 6\frac{n^2}{\ell^2} + 2\nu^2 \right) . \quad (52)$$

Thus the entropy product formula for dyonic Taub-NUT is

$$S_+ S_- = S_+^2 = S_-^2 = (4\pi n^2)^2 \left(1 - 6\frac{n^2}{\ell^2} + 2\nu^2 \right)^2 . \quad (53)$$

The product formula is indeed independent of mass and depends on n , ℓ and ν .

For a dyonic Bolt, we set $r_{\pm} = r_b$ and satisfies the fourth order equation for r_b :

$$6nr_b^4 - \ell^2 r_b^3 + 2n (\ell^2 - 3n^2 + \ell^2 \nu^2) r_b^2 - 2\ell^2 \nu^2 n^3 = 0 . \quad (54)$$

Let $r_b = r_{b\pm}$ be the solution of the equation. Then we find the action as previously

$$I_{\pm} = -2\pi \frac{[r_b^4 - \ell^2 r_b^2 + n^2 (3n^2 - \ell^2)] r_b^2 - (r_b^4 + 4n^2 r_b^2 - n^4) \ell^2 \nu^2}{(3r_b^2 - 3n^2 + \ell^2) r_b^2 + (r_b^2 - n^2) \ell^2 \nu^2} . \quad (55)$$

Similarly the entropy is given by

$$S_{\pm} = 2\pi \frac{[3r_b^4 + (\ell^2 - 12n^2) r_b^2 + n^2 (\ell^2 - 3n^2)] r_b^2 + (r_b^4 + 4n^2 r_b^2 - n^4) \ell^2 \nu^2}{(3r_b^2 - 3n^2 + \ell^2) r_b^2 + (r_b^2 - n^2) \ell^2 \nu^2} . \quad (56)$$

After substituating the value of $r_{b\pm}$ in the entropy product formula, it seems that the product is independent of mass and depends on n, ℓ, ν for dyonic Taub-Bolt instanton.

9 Conclusion:

We have studied the mass-independent feature for various gravitational instantons. This universal feature gives us strong indication to understanding the microscopic properties of BH entropy. It would be interesting if one considered the entropy product formula for other instantons like Multi-Taub NUT, Non-self dual Taub-NUT, S^4 , CP^2 , $S^2 \times S^2$ and Twisted $S^2 \times S^2$. We expect these instantons also gives us universal features.

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